



ANIQ INTEGRALLARNI HISOBLASH

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Annotatsiya : Maqola olimpiada tanlovlaridagi aniq integrallarni hisoblashning umumiy va xususiy hollariga qaratilgan. Maqolada toq funksiyalar orqali hisoblash, integralni integrallar yig‘indisiga ajratish bilan, integralni yangi o‘zgaruvchi kiritish usullari bilan hisoblash ko‘rsatilgan.

Tayanch so‘zlar: Aniq integral, Nyuton-Leybnis formulasi, toq funksiya, differensial, chegaralangan, ko‘phad, ayniyat, umumiy ko‘paytuvchi, argument, yechim.

KIRISH

Aniq integral ostidagi funksiya, toq funksiya bo‘lsa, u holda

$$\int_{-a}^a f(x)dx = 0, \quad f(-x) = -f(x), \quad (1)$$

tenglikdan foydalanish mumkin.

$$\int_{-1}^1 x^2 \ln(x + \sqrt{x^2 + 1}) dx$$

1-misol. aniq integralni hisoblang.

Yechish. Bu integral ostidagi funksiya toq funksiya ekanligini ko‘rsatib hisoblaymiz.

$$\begin{aligned} f(-x) &= (-x)^2 \ln(-x + \sqrt{(-x)^2 + 1}) = x^2 \ln\left(\frac{(\sqrt{x^2 + 1} - x)(\sqrt{x^2 + 1} + x)}{\sqrt{x^2 + 1} + x}\right) = \\ &= x^2 \ln\left(\frac{x^2 + 1 - x^2}{\sqrt{x^2 + 1} + x}\right) = x^2 \ln\left(\frac{1}{\sqrt{x^2 + 1} + x}\right) = -x^2 \ln(x + \sqrt{x^2 + 1}) = -f(x) \end{aligned}$$

Integral ostidagi funksiyaning toqligidan, $\int_{-1}^1 x^2 \ln(x + \sqrt{x^2 + 1}) dx = 0$, ga teng.

2-misol. $\int_0^{2008} x \cdot (x-4) \cdot (x-8) \dots (x-2008) \cdot dx$ integralni hisoblang.

Yechish. Aniq integralni hisoblash uchun, quyidagicha belgilash kiritib, yuqoridagi ⁽¹⁾ dan foydalanib, toq funksiyaga keltiramiz.

$$\begin{aligned} \int_0^{2008} x \cdot (x-4) \cdot (x-8) \dots (x-2008) \cdot dy &= \left| \begin{array}{l} y = x - 1004 \\ dx = dy \\ -1004 \leq y \leq 1004 \end{array} \right| = \\ &= \int_{-1004}^{1004} (y+1004) \cdot (y+1000) \dots (y+4) \cdot y \cdot (y-4) \dots (y-1000) \cdot (y-1004) \cdot dy = \\ &= \int_{-1004}^{1004} (y^2 - 1004^2) \cdot (y^2 - 1000^2) \dots (y^2 - 4^2) \cdot y \cdot dy = 0 \end{aligned}$$

Integralning qiymati nolga teng ekan.

3-misol. $\int_{-a}^a \frac{1}{1 + \varphi(x) + \sqrt{1 + \varphi^2(x)}} dx$, integralni hisoblang, bu yerda $\varphi(-x) = -\varphi(x)$ ga teng.

Yechish. Aniq integralni hisoblashda berilgan shartlardan foydalanib hisoblaymiz.

$$\begin{aligned}
 \int_{-a}^a \frac{1}{1+\varphi(x)+\sqrt{1+\varphi^2(x)}} dx &= \int_{-a}^0 \frac{1}{1+\varphi(x)+\sqrt{1+\varphi^2(x)}} dx + \int_0^a \frac{1}{1+\varphi(x)+\sqrt{1+\varphi^2(x)}} dx = \\
 &= \int_0^a \frac{1}{1-\varphi(x)+\sqrt{1+\varphi^2(x)}} dx + \int_0^a \frac{1}{1+\varphi(x)+\sqrt{1+\varphi^2(x)}} dx = \\
 &= \int_0^a \left(\frac{1}{1+\sqrt{1+\varphi^2(x)}-\varphi(x)} + \frac{1}{1+\sqrt{1+\varphi^2(x)}+\varphi(x)} \right) dx = \\
 &= \int_0^a \left(\frac{1+\sqrt{1+\varphi^2(x)}+\varphi(x)+1+\sqrt{1+\varphi^2(x)}-\varphi(x)}{(1+\sqrt{1+\varphi^2(x)})^2-\varphi^2(x)} \right) dx = \int_0^a \frac{2+2\sqrt{1+\varphi^2(x)}}{2+2\sqrt{1+\varphi^2(x)}} dx = \int_0^a 1 dx = a.
 \end{aligned}$$

4-misol. $\int_{-1}^1 \frac{1}{1+x^3+\sqrt{1+x^6}} dx$ integralni hisoblang.

Yechish. Bu yerda, birinchi qo’shiluvchida $x = -x$ deb olib ikkinchi integral o’zini yozib, quyidagicha hisoblaymiz.

$$\begin{aligned}
 \int_{-1}^1 \frac{1}{1+x^3+\sqrt{1+x^6}} dx &= \int_{-1}^0 \frac{1}{1+x^3+\sqrt{1+x^6}} dx + \int_0^1 \frac{1}{1+x^3+\sqrt{1+x^6}} dx = \\
 &= \int_0^1 \frac{1}{1-x^3+\sqrt{1+x^6}} dx + \int_0^1 \frac{1}{1+x^3+\sqrt{1+x^6}} dx = \\
 &= \int_0^1 \left(\frac{1}{1-x^3+\sqrt{1+x^6}} + \frac{1}{1+x^3+\sqrt{1+x^6}} \right) dx = \\
 &= \int_0^1 \left(\frac{1+x^3+\sqrt{1+x^6}+1-x^3+\sqrt{1+x^6}}{(1+\sqrt{1+x^6})^2-x^6} \right) dx = \int_0^1 \frac{2+2\sqrt{1+x^6}}{2+2\sqrt{1+x^6}} dx = \int_0^1 1 dx = 1.
 \end{aligned}$$

Ma’lumki, $\frac{1}{1+a} + \frac{1}{1+a^{-1}} = \frac{1}{1+a} + \frac{a}{1+a} = 1$ tenglikdan ham foydalanib integralni oson hisoblash mumkin.

5-misol. $\int_{-1}^1 \frac{1}{(e^x + 1)(1 + x^2)} dx$ integralni hisoblang.

Yechish. Yuqoridagi $\frac{1}{1+a} + \frac{1}{1+a^{-1}} = 1$ ayniyatdan, foydalanib, integralni qiymatini topamiz.

$$\begin{aligned} \int_{-1}^1 \frac{1}{(e^x + 1)(1 + x^2)} dx &= \int_{-1}^0 \frac{1}{(e^{-x} + 1)(1 + x^2)} dx + \int_0^1 \frac{1}{(e^x + 1)(1 + x^2)} dx = \\ &= \int_0^1 \left(\frac{1}{(e^{-x} + 1)(1 + x^2)} + \frac{1}{(e^x + 1)(1 + x^2)} \right) dx = \int_0^1 \left(\frac{1}{e^{-x} + 1} + \frac{1}{e^x + 1} \right) \cdot \frac{1}{1 + x^2} dx = \\ &= \int_0^1 \left(\frac{e^x}{e^x + 1} + \frac{1}{e^x + 1} \right) \cdot \frac{1}{1 + x^2} dx = \int_0^1 \frac{1}{1 + x^2} dx = \arctg x \Big|_0^1 = \arctg 1 - \arctg 0 = \frac{\pi}{4}. \end{aligned}$$

6-misol. $\int_{-2019}^{2019} \frac{1}{1 + 2^{\pi \operatorname{tg} x}} dx$ integralni hisoblang.

Yechish. Berilgan integralni quyidagicha yozib hisoblash mumkin.

$$\begin{aligned} \int_{-2024}^{2024} \frac{1}{1 + 2^{\pi \operatorname{tg} x}} dx &= \int_{-2024}^0 \frac{1}{1 + 2^{\pi \operatorname{tg} x}} dx + \int_0^{2024} \frac{1}{1 + 2^{\pi \operatorname{tg} x}} dx = \int_0^{2024} \frac{1}{1 + 2^{-\pi \operatorname{tg} x}} dx + \int_0^{2024} \frac{1}{1 + 2^{\pi \operatorname{tg} x}} dx = \\ &= \int_0^{2024} \left(\frac{1}{1 + 2^{\pi \operatorname{tg} x}} + \frac{1}{1 + 2^{-\pi \operatorname{tg} x}} \right) dx = \int_0^{2024} \left(\frac{2 + 2^{-\pi \operatorname{tg} x} + 2^{\pi \operatorname{tg} x}}{2 + 2^{-\pi \operatorname{tg} x} + 2^{\pi \operatorname{tg} x}} \right) dx = \int_0^{2024} 1 dx = 2024 \end{aligned}$$

7-misol. $\int_0^{\frac{\pi}{2}} (\cos^2(\cos x) + \sin^2(\sin x)) dx$ integralni hisoblang.

Yechish. Bu integralni hisoblashda $I = \int_0^{\frac{\pi}{2}} (\cos^2(\cos x) + \sin^2(\sin x)) dx$ deb olib,

$x = \frac{\pi}{2} - y$ belgilash kiritib olamiz, bu yerda

$$\sin\left(\frac{\pi}{2} - y\right) = \cos y, \quad \cos\left(\frac{\pi}{2} - y\right) = \sin y$$

ga teng.

$$I = \int_0^{\frac{\pi}{2}} \left(\cos^2(\cos x) + \sin^2(\sin x) \right) dx = \left| \begin{array}{l} x = \frac{\pi}{2} - y \\ dx = -dy \\ \frac{\pi}{2} \leq y \leq 0 \end{array} \right| = \int_0^{\frac{\pi}{2}} \left(\cos^2\left(\cos\left(\frac{\pi}{2} - y\right)\right) + \sin^2\left(\sin\left(\frac{\pi}{2} - y\right)\right) \right) dy =$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos^2(\sin y) + \sin^2(\cos y) \right) dy = \int_0^{\frac{\pi}{2}} \left(1 - \sin^2(\sin y) + 1 - \cos^2(\cos y) \right) dy =$$

$$= \int_0^{\frac{\pi}{2}} 2dy - \int_0^{\frac{\pi}{2}} \left(\sin^2(\sin y) + \cos^2(\cos y) \right) dy$$

Berilgan integral $\int_0^{\frac{\pi}{2}} \left(\cos^2(\cos x) + \sin^2(\sin x) \right) dx$ va $\int_0^{\frac{\pi}{2}} \left(\sin^2(\sin y) + \cos^2(\cos y) \right) dy$

larning qiymati o'zgaruvchiga bog'liq emas, demak ularning qiymatlari bir xil ya'ni

$$I = \int_0^{\frac{\pi}{2}} 2dy - I, \Rightarrow 2I = \pi, \Rightarrow I = \frac{\pi}{2}.$$

I ga teng. Oxirgi tenglikdan

Berilgan integralni qiymati $\frac{\pi}{2}$ ga teng.

XULOSA

Maqolada olimpiada tanlovlaridagi aniq integrallarni hisoblashning umumiy va xususiy hollari keltirilgan. Toq funksiyalar orqali hisoblash, integralni integrallar yig'indisiga ajratish yoki yangi o'zgaruvchi kiritish usullari bilan hisoblash yo'llari ko'rsatilgan. Yuqoridagi usullardan foydalanib aniq integrallarni hisoblash, o'ylaymizki, matematika ixlosmandlariga foydali bo'ladi.

FOYDALANILGAN ADABIYOTLAR

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